

Chapter Vectors and Matrices

§ Vectors

Row vector $\bar{u} = (u_1, \dots, u_n) \in R^n$, Column vector $\bar{u} = (u_1, \dots, u_n)^T \in R^n$

Two row vectors $\bar{u}$ and $\bar{v} \in R^n$, $\alpha, \beta \in R$

$$\alpha\bar{u} + \beta\bar{v} = (\alpha u_1 + \beta v_1, \dots, \alpha u_n + \beta v_n) \in R^n$$

$$\text{Inner product } \bar{u} \bullet \bar{v} = \bar{u}\bar{v}^T = \bar{v} \bullet \bar{u} = \bar{v}^T \bar{u} = \sum_{i=1}^n u_i v_i$$

$$\text{Length of } \bar{u} \text{ is } \|\bar{u}\| = (\bar{u}\bar{u}^T)^{1/2}$$

Let $\bar{u}$ and $\bar{v} \in C^n$, then the conjugate of $\bar{u}$ is denoted as $\bar{\bar{u}}$

The hermitian inner product of $\bar{u}$ into $\bar{v}$ is defined as $\bar{\bar{u}} \bullet \bar{v}$

The hermitian length of $\bar{u}$ is defined as $\bar{\bar{u}} \bullet \bar{u}$.

If $\sum_{i=1}^m \alpha_i \bar{u}_i = \bar{0}$ has only one solution $\alpha_i = 0$, $i = 1 \sim m$, then $\bar{u}_i$, $i = 1 \sim m$,

are **linearly independent**. If at least one $\alpha_i \neq 0$, then $\bar{u}_i$, $i = 1 \sim m$, are

linearly dependent. If $\bar{u}_i$, $i = 1 \sim n$, are linearly independent, then the

expression $\bar{x} = \sum_{i=1}^n \alpha_i \bar{u}_i$ is unique for $\bar{x} \in R^n$. In this situation, R^n is

spanned by $\bar{u}_i$, $i = 1 \sim n$, and the set of $\bar{u}_i$, $i = 1 \sim n$, is said to form a

basis for R^n .

Gram-Schmidt procedure

Given a set of linearly independent vectors $\bar{u}_i$, $i = 1 \sim n$, to generate a set

of orthonormal vectors $\bar{v}_i$, $i = 1 \sim n$, i.e., $\bar{v}_i^T \bar{v}_j = \delta_{ij}$, for $i \neq j$.

$$\bar{v}_1 = \bar{u}_1 / \|\bar{u}_1\|, \quad \bar{w}_i = \bar{u}_i - \sum_{j=1}^{i-1} (\bar{u}_i^T \bar{v}_j) \bar{v}_j, \quad \bar{v}_i = \bar{w}_i / \|\bar{w}_i\|.$$

§ Matrices

$$A = \begin{bmatrix} a_{11} & \bullet & \bullet & a_{1m} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ a_{n1} & \bullet & \bullet & a_{nm} \end{bmatrix} \in R^{n \times m}, \quad \left\{ \begin{bmatrix} a_{1j} \\ \bullet \\ \bullet \\ a_{nj} \end{bmatrix} \right\} \text{ is the } j\text{th column of } A$$

$\{a_{i1} \quad \bullet \quad \bullet \quad a_{im}\}$ is the i th row of A .

The **column space** of $A \in R^{m \times n}$ is the space spanned by the columns of A and is denoted as $R[A] = \{A\bar{x} \mid \bar{x} \in R^n\}$. The **row space** of A is the space spanned by the row of A and is denoted as $R[A^T] = \{\bar{x}^T A \mid \bar{x} \in R^m\}$

The right null space of A is denoted as $N[A] = \{\bar{x} \mid A\bar{x} = \bar{0} \in R^m\}$

The left null space of A is denoted as $N[A^T] = \{\bar{x} \mid \bar{x}^T A = \bar{0} \in R^n\}$ or

$$N[A^T] = \{\bar{x} \mid A^T \bar{x} = \bar{0} \in R^n\}.$$

Rank

$r \triangleq \text{rank } A = \text{dimension of column space } R[A]$

$\dim N[A] \triangleq \text{dimension of null space of } N[A]$

$n \triangleq \text{total number of columns of } A$

$$r + \dim N[A] = n.$$

$r \triangleq \text{rank } A = \text{dimension of row space } R[A^T]$

$\dim N[A^T] \triangleq \text{dimension of left null space of } N[A^T]$

$k \triangleq \text{total number of rows of } A$

$$r + \dim N[A^T] = k.$$

Theorem.

Matrix

$N[A]$ and $R[A^*]$ are orthogonal subspace. The fact is denoted as

$$R[A^T]^\perp = N[A]$$

Proof:

$\forall y \in N[A]$ and $z \in R[A^T]$ imply there exists x such that $A^T x = z$ and

$$Ay = 0, \text{ then } z^T y = (A^T x)^T y = x^T Ay = 0 \text{ or } N[A] \perp R[A^T].$$

Theorem

$N[A^T]$ and $R[A]$ are orthogonal subspace. The fact is denoted as

$$R[A]^\perp = N[A^T]$$

Proof:

$\forall y \in N[A^T]$ and $z \in R[A]$ imply there exists x such that $Ax = z$ and

$$A^T y = 0, \text{ then } z^T y = (Ax)^T y = x^T A^T y = 0 \text{ or } N[A^T] \perp R[A].$$

The norm of $x \in R^n$ is defined as $\|x\|^2 = x^T x = \sum_{i=1}^n x_i^2$

The problem $\|Ax\|^2 = x^T A^T Ax = 0$ will imply $x = 0$ if the columns of A are linearly independent or $A^T A$ is non-singular.

Consider the least square problem $Ax = y$, $A \in R^{n \times k}$, $x \in R^k$, $y \in R^n$,

$$n \leq k. \quad A^T Ax = A^T y \text{ implies } x = (A^T A)^{-1} A^T y.$$

$$\|Ax\|^2 = x^T A^T Ax = 0$$

$$A = [a_{ij}]_{n \times m}, \quad B = [b_{ij}]_{n \times m}, \quad \alpha A + \beta B = [\alpha a_{ij} + \beta b_{ij}]_{n \times m}$$

$$A = [a_{ij}]_{n \times m}, \quad C = [c_{ij}]_{m \times p}, \quad AC = \left[\sum_{j=1}^m a_{ij} c_{jk} \right]_{n \times p}$$

Transpose of A is denoted as A^T , $A^T = \begin{bmatrix} a_{11} & \bullet & \bullet & a_{n1} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ a_{1m} & \bullet & \bullet & a_{nm} \end{bmatrix} \in R^{m \times n}$, $a_{ji}^T = a_{ij}$

The j th column of A becomes the j th row of A^T .

The i th row of A becomes the i th column of A^T .

If $A = A^T$, then A is symmetric. If $A = -A^T$, then A is skew-symmetric

The conjugate of A is denoted as $\bar{A}$

The hermitian conjugate of A is denoted as $A^* (= \bar{A}^T)$

$A \in R^{n \times n}$ is an identity matrix if $a_{ii} = 1$, $i = 1 \sim n$, and others being zero

Properties

1. $(A^T)^T = A$, 2. $(A + B)^T = A^T + B^T$, 3. $(AB)^T = B^T A^T$

§ Determinants

$A \in R^{n \times n}$, the determinant of A is denoted as $|A|$ or $\det(A)$

Deleting the i th row and the j th column of A yields a new matrix

$$A^{(1)} \in R^{(n-1) \times (n-1)}$$

The determinant of $A^{(1)}$ is called the minor M_{ij} of a_{ij} . The value

$$A_{ij} = (-1)^{i+j} M_{ij}$$

is called the cofactor of a_{ij} .

$$\text{Ex. } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The minor M_{22} and cofactor A_{22} of a_{22} are $M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$ and

$A_{22} = (-1)^{2+2} M_{22}$, respectively.

Properties

1. $\det(A) = \sum_{k=1}^n a_{ik} A_{ik} = \sum_{k=1}^n a_{kj} A_{kj}$, 2. $\det(A) = \det(A^T)$
3. If A is a triangular matrix, then $\det(A) = \prod_{i=1}^n a_{ii}$
4. If A has either a zero row vector or a zero column vector, then $\det(A) = 0$
5. $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \vec{f}_i + \vec{g}_i \dots \vec{v}_n]$, then $\det(A) = \det[\vec{v}_1 \ \vec{v}_2 \ \dots \vec{f}_i \dots \vec{v}_n] + \det[\vec{v}_1 \ \vec{v}_2 \ \dots \vec{g}_i \dots \vec{v}_n]$
6. If A has either the same two row vectors or the same two column vectors, then $\det(A) = 0$
7. If B is a matrix obtained by interchanging any two rows (columns) of a square matrix, then $\det(B) = -\det(A)$
8. If one row (column) vector of a square matrix A is equal to a number c times some other row (column) vector, then $\det(A) = 0$
9. $A = [\vec{v}_1 \ \vec{v}_2 \ \dots \vec{v}_i \dots \vec{v}_n]$, $B = [\vec{v}_1 \ \vec{v}_2 \ \dots c\vec{v}_i \dots \vec{v}_n]$, then $\det(B) = c \det(A)$
10. $\det(cA) = c^n \det(A)$
11. $\sum_{k=1}^n a_{ik} A_{jk} = \sum_{k=1}^n a_{kj} A_{ki} = \delta_{ij} \det(A)$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
12. $A, B \in R^{n \times n}$, $\det(AB) = \det(A) \det(B)$
13. If $f_1(x) \sim f_3(x)$ are differentiable, then

$$\frac{d}{dx} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$$

Hw

1. Show that $\det(A) = \begin{vmatrix} b^2 + ac & bc & c^2 \\ ab & 2ac & bc \\ a^2 & ab & b^2 + ac \end{vmatrix} = 4a^2b^2c^2$

2. Find the value $\begin{vmatrix} b^2 + c^2 & bc & ac \\ ab & c^2 + a^2 & bc \\ ca & bc & b^2 + a^2 \end{vmatrix}$

3. Find the value of k such that $\begin{vmatrix} k & 3+k & -10 \\ 1-k & 2-k & 5 \\ 2 & 4+k & -k \end{vmatrix} = 48$

4. $\det \begin{vmatrix} \cos t & \sin t & \ln t \\ -\sin t & \cos t & 1/t \\ -\cos t & -\sin t & -1/t^2 \end{vmatrix}$

§ Inverse

$A, B \in R^{n \times n}$ If $AB = BA = I$, then $B = A^{-1}$

From $\sum_{k=1}^n a_{ik} A_{jk} = \sum_{k=1}^n a_{kj} A_{ki} = \delta_{ij} \det(A)$, we obtain

$$\begin{bmatrix} a_{11} & \bullet & \bullet & a_{1n} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ a_{n1} & \bullet & \bullet & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \bullet & \bullet & A_{n1} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ A_{1n} & \bullet & \bullet & A_{nn} \end{bmatrix} = \begin{bmatrix} |A| & 0 & \bullet & 0 \\ 0 & |A| & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & |A| \end{bmatrix}$$

Define the adjoint of A as $\text{adj}(A) = [A_{ij}]^T$, then

$$A \text{adj}(A) = |A| I, \text{ and } A^{-1} = \text{adj}(A) / |A|$$

Ex. $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 3 \\ 3 & 1 & -2 \end{bmatrix}$, $A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} = -3$, $A_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix} = 7$,

$$A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} = -1, \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = 8, \quad A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = -14,$$

Matrix

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 5, \quad A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = 6, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} = -7,$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 2,$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \times (-3) + 2 \times 7 + 4 \times (-1) = 7$$

$$\text{adj}(A) = \begin{bmatrix} -3 & 7 & -1 \\ 8 & -14 & 5 \\ 6 & -7 & 2 \end{bmatrix}^T = \begin{bmatrix} -3 & 8 & 6 \\ 7 & -14 & -7 \\ -1 & 5 & 2 \end{bmatrix}$$

The Problem $A\bar{x} = \bar{b}$ can be solved as $\bar{x} = A^{-1}\bar{b}$

$$\text{Or } \begin{Bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{Bmatrix} = \frac{1}{|A|} \text{adj}(A) \begin{Bmatrix} b_1 \\ \bullet \\ \bullet \\ b_n \end{Bmatrix} = \frac{1}{|A|} \begin{bmatrix} A_{11} & \bullet & \bullet & A_{n1} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ A_{1n} & \bullet & \bullet & A_{nn} \end{bmatrix} \begin{Bmatrix} b_1 \\ \bullet \\ \bullet \\ b_n \end{Bmatrix} = \frac{1}{|A|} \begin{Bmatrix} \sum_{i=1}^n b_i A_{i1} \\ \bullet \\ \bullet \\ \sum_{i=1}^n b_i A_{in} \end{Bmatrix}$$

Cramer's rule

$$\text{Ex. } \begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 4 \\ 6 & 8 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \\ 5 \end{Bmatrix},$$

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 9 & 5 & 4 \\ 6 & 8 & 7 \end{bmatrix} = 1 \times 5 \times 7 + 9 \times 8 \times 3 + 6 \times 4 \times 2 - 3 \times 5 \times 6 - 4 \times 8 \times 1 - 7 \times 9 \times 2$$

$$= 35 + 216 + 48 - 90 - 32 - 126 = 35 + 48 - 32 = 51$$

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 4 \\ 5 & 8 & 7 \end{bmatrix} = 35 + 72 + 40 - 75 - 32 - 42 = 72 - 32 - 42 = -2$$

$$\det \begin{bmatrix} 1 & 1 & 3 \\ 9 & 3 & 4 \\ 6 & 5 & 7 \end{bmatrix} = 21 + 135 + 24 - 54 - 20 - 63 = 43$$

Matrix

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 9 & 5 & 3 \\ 6 & 8 & 5 \end{bmatrix} = 25 + 72 + 36 - 30 - 24 - 90 = -11$$

$$(x_1, x_2, x_3) = (-2, 43, -11) / 51$$

A is non-singular if $|A| \neq 0$

If A and B are non-singular, then $(AB)^{-1} = B^{-1}A^{-1}$, $A^{-k} = (A^{-1})^k = (A^k)^{-1}$

Hw

1. Find the adjoint and inverse of matrix A

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

2. Solve the following equation by Cramer's rule

$$x - y + 2z = -5$$

$$-x + 3z = 1$$

$$2x + y = 3$$

3. If $|A| = 3$, $A \in R^{n \times n}$, then what are the values $|A^{-1}|$, $|adj(A)|$ and $|adj(A^{-1})|$?

§ Eigenvalues

$A \in R^{n \times n}$, $A\bar{x} = \lambda\bar{x}$, λ is the eigenvalue and $\bar{x}$ is the eigenvector

Consider $\frac{dy}{dx} - \lambda y = 0$. Let $y = ae^{\alpha x}$, then $a\alpha e^{\alpha x} = a\lambda e^{\alpha x}$ or $\alpha = \lambda$.

Consider $\frac{d\bar{y}}{dx} - A\bar{y} = 0$. Let $\bar{y} = e^{\alpha x}\bar{u}$, then $\alpha\bar{u} = \lambda A\bar{u}$

$A\bar{x} = \lambda\bar{x}$ implies $(A - \lambda I)\bar{x} = \bar{0}$. Since $\bar{x} \neq \bar{0}$, then $A - \lambda I$ is singular or

$$|A - \lambda I| = 0.$$

Matrix

$$\text{Ex. } A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}, \quad \det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & -5 \\ 1 & -2 - \lambda \end{bmatrix} = 5 - (4 - \lambda)(2 + \lambda) = 0$$

$$5 + (\lambda - 4)(2 + \lambda) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \quad \begin{bmatrix} 1 & -5 \\ 1 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix}$$

$$\lambda = -1, \quad \begin{bmatrix} 5 & -5 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Hw

(a) Find all real values of k for which the matrix

$$A = \begin{bmatrix} 1 & k & 3 \\ -k & 2 & -k \\ 1 & k & 3 \end{bmatrix}$$

has real eigenvalues

(b) Find the corresponding eigenvector of A

(c) With k real, determine the largest possible eigenvalue of A and the corresponding eigenvector

(d) Find the real values of k for which A has only two real eigenvalues.

Find the two eigenvalues and the corresponding eigenvectors.

Theorem

A matrix is singular if and only if at least one of its eigenvalues is zero.

Theorem

If A and B are similar matrices, then A and B have the same eigenvalues.

Remark. A and B are similar matrices, if $B = S^{-1}AS$.

Theorem

A eigenvector of a square matrix cannot correspond to two distinct eigenvalues.

Theorem

If $\bar{x}_1 \dots \bar{x}_n$ are eigenvectors corresponding respectively to distinct eigenvalues $\lambda_1 \dots \lambda_n$, then $\bar{x}_1 \dots \bar{x}_n$ are linearly independent.

Theorem

The eigenvalues of a hermitian matrix are all real.

Corollary

The eigenvalues of a real symmetric matrix are all real.

Corollary

The eigenvalues of a skew-hermitian matrix are all pure imaginary.

Theorem

If $\bar{x}_i$ and $\bar{x}_j$ are eigenvectors corresponding, respectively, to the distinct eigenvalues λ_i and λ_j of a hermitian matrix H , then $\bar{x}_i^T \bar{x}_j = 0$ and $\bar{x}_i^T H \bar{x}_j = 0$.

Theorem

Every $n \times n$ hermitian (real symmetric) matrix has n linearly independent eigenvectors.

Corollary

Every $n \times n$ hermitian (real symmetric) matrix has a set of n orthonormal eigenvectors.

Theorem

The equation $(A - \lambda B)\bar{x} = \bar{0}$ has zero as an eigenvalue iff A is singular.

Theorem

If A and B are hermitian (or real symmetric) matrices, and if B is

definite, then the eigenvalues of $(A - \lambda B)\bar{x} = \bar{0}$ are real.

Corollary

If A and B are hermitian (or real symmetric) matrices, and if B is positive-definite and A is negative-definite, or vice versa, then all the eigenvalues of $(A - \lambda B)\bar{x} = \bar{0}$ are negative.

Corollary

If A and B are hermitian (or real symmetric) matrices, and if both B and A are positive-definite or negative-definite, then all the eigenvalues of $(A - \lambda B)\bar{x} = \bar{0}$ are positive.

Theorem

If A and B are hermitian (or real symmetric) matrices, and if $\bar{x}_1, \dots, \bar{x}_n$ are eigenvectors of the equation $(A - \lambda B)\bar{x} = \bar{0}$ corresponding, respectively, to the distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then $\bar{x}_i^* A \bar{x}_j = 0$ and $\bar{x}_i^* B \bar{x}_j = 0$ for $i \neq j$.

Theorem

If A and B are hermitian (or real symmetric) matrices, if B is definite, and if $\bar{x}_1, \dots, \bar{x}_n$ are eigenvectors of the equation $(A - \lambda B)\bar{x} = \bar{0}$ corresponding, respectively, to the distinct eigenvalues $\lambda_1, \dots, \lambda_n$, then $\bar{x}_1, \dots, \bar{x}_n$ are linearly independent.

Theorem

If A and B are hermitian (or real symmetric) matrices, and if B is positive-definite, then all eigenvectors of the equation $(A - \lambda B)\bar{x} = \bar{0}$ are linearly independent and orthonormal with respect to B .

Ex. Both A and B are non-singular, then AB and BA have the same

eigenvalues.

Ex. All eigenvalues of a skew-hermitian matrix are pure imaginary or zero.

Ex. Let $\bar{x}_1 \dots \bar{x}_n$ be orthonormal eigenvectors of the equation $(A - \lambda I)\bar{x} = \bar{0}$ corresponding the eigenvectors $\lambda_1 \dots \lambda_n$, respectively. Then the solution of $(A - \lambda I)\bar{x} = \bar{v}$ is given by

$$\bar{x} = \sum_{i=1}^n \left(\frac{\bar{x}_i^T \bar{v}}{\lambda_i - \lambda} \right) \bar{x}_i$$

Theorem

Let $(\lambda_i, \bar{x}_i)$ be the i th eigenvalue and associated eigenvector, $i = 1 \sim n$, of the equation $(A - \lambda B)\bar{x} = \bar{0}$, in which both A and B are hermitian, and B is positive-definite. Let $\bar{x}_i^* B \bar{x}_j = \delta_{ij}$ and $M = [\bar{x}_1 \dots \bar{x}_n]$, then

$$M^* B M = I, \quad M^* A M = \text{diagonal matrix of } [\lambda].$$

Theorem

An $n \times n$ matrix is similar to a diagonal matrix iff it has n linearly independent eigenvectors.

Remark: A is similar to B if $A = S^{-1} B S$

Corollary

Every hermitian matrix and every real symmetric matrix is similar to a diagonal matrix.

§ Functions of a square matrix

$$A^r A^s = A^{r+s} = A^{s+r} = A^s A^r, \quad A^{-s} = (A^s)^{-1} = (A^{-1})^s$$

Let $f(A)$ and $g(A)$ be two polynomials of A ,

then $f(A)g(A) = g(A)f(A)$.

If $g(A)$ is non-singular, then $f(A)g^{-1}(A) = g^{-1}(A)f(A) = f(A)/g(A)$.

Theorem

Let $\lambda_1 \dots \lambda_n$ be eigenvalues of A . If f is a polynomial, then

$$|f(A)| = f(\lambda_1) \dots f(\lambda_n)$$

Proof: Express $f(x)$ as $f(x) = c(x - b_1) \dots (x - b_k)$, then

$$f(A) = c(A - b_1 I) \dots (A - b_k I),$$

$$\begin{aligned} |f(A)| &= c^k |A - b_1 I| \dots |A - b_k I| = c \prod_{j=1}^k \prod_{i=1}^n (\lambda_i - b_j) = \prod_{i=1}^n \left(c \prod_{j=1}^k (\lambda_i - b_j) \right) \\ &= \prod_{i=1}^n f(\lambda_i) \end{aligned}$$

Theorem

Let $\lambda_1 \dots \lambda_n$ be eigenvalues of A . If f is a polynomial, then the eigenvalues of $f(A)$ are $f(\lambda_1), \dots$, and $f(\lambda_n)$

Prof.

Let $g(x) = f(x) - \lambda = 0$, then $|g(A)| = |f(A) - \lambda I| = |f(\lambda_1) - \lambda| \dots |f(\lambda_n) - \lambda| = 0$

So, $\lambda = f(\lambda_1), \dots, f(\lambda_n)$.

Corollary

Let λ and $\bar{x}$ be one eigenvalue and the associated eigenvector of A , then

$f(\lambda)$ and $\bar{x}$ are one eigenvalue and the associated eigenvector of $f(A)$.

Theorem

If A is a matrix which is similar to a diagonal matrix, that is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}, \text{ then } p(A) = S \begin{bmatrix} p(\lambda_1) & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & p(\lambda_n) \end{bmatrix} S^{-1}$$

$$\text{Ex. } p(x) = x^4 - 4x^3 + 6x^2 - x - 3, \quad A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \det \begin{bmatrix} -\lambda & -2 \\ 1 & 3 - \lambda \end{bmatrix} = 0 = 2 + \lambda(-3 + \lambda), \quad \lambda_1 = 1, \quad \lambda_2 = 2$$

$$\lambda_1 = 1: \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

$$\lambda_2 = 2: \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}^{-1}$$

$$p(A) = S \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} S^{-1}$$

Theorem

If A is similar to a diagonal matrix, and if p is a scalar polynomial, then the equation $p(X) = A$ is solvable for X .

$$\text{Ex. Solve } X^2 - 4X + 4I = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}$$

$$\det \begin{bmatrix} 4 - \lambda & 3 \\ 5 & 6 - \lambda \end{bmatrix} = 0 = (\lambda - 4)(\lambda - 6) - 15 = \lambda^2 - 10\lambda + 9, \quad \lambda_1 = 1, \quad \lambda_2 = 9$$

$$\lambda_1 = 1: \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Matrix

$$\lambda_2 = 9: \begin{bmatrix} -5 & 3 \\ 5 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 5 \end{Bmatrix}, \quad S = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} S^{-1}.$$

$$\text{Let } Y = SXS^{-1}, \quad Y^2 - 4Y + 4I = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}, \quad y_1^2 - 4y_1 + 4 = 1, \quad y_1 = 1, 3$$

$$y_2^2 - 4y_2 + 4 = 9, \quad y_2 = -1, 5$$

$$Y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}, \quad X = S^{-1}YS$$

$$\text{Ex. } A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix},$$

$$e^A = S \begin{bmatrix} e & 0 \\ 0 & e^9 \end{bmatrix} S^{-1}, \quad \sin(A) = S \begin{bmatrix} \sin(1) & 0 \\ 0 & \sin(9) \end{bmatrix} S^{-1},$$

$$\sin^2 A + \cos^2 A = S \begin{bmatrix} \sin^2(1) + \cos^2(1) & 0 \\ 0 & \sin^2(9) + \cos^2(9) \end{bmatrix} S^{-1} = I$$

$$0 = |A - \lambda I| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0 \quad \text{implies} \quad A^n + c_{n-1}A^{n-1} + \dots + c_0I = 0$$

$$\text{where } c_0 = |A|. \quad A^n = -c_{n-1}A^{n-1} - \dots - c_0I.$$

$$\text{If } A^{-1} \text{ exist, then } A^{n-1} + c_{n-1}A^{n-2} + \dots + c_0A^{-1} = 0 \quad \text{or}$$

$$A^{-1} = -(A^{n-1} + c_{n-1}A^{n-2} + \dots + c_1I) / \det(A)$$

§ System of Differential Equations

$$\text{Ex. } \frac{dx}{dt} + 2x + \frac{dy}{dt} + 6y = 2e^t, \quad 2\frac{dx}{dt} + 3x + 3\frac{dy}{dt} + 8y = -1$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = e^t \begin{Bmatrix} 2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{Bmatrix} 2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -4 \end{Bmatrix}, \quad \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\frac{d}{dt} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = e^t \begin{Bmatrix} 6 \\ -4 \end{Bmatrix} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Eigenvalues $\det \begin{bmatrix} 3-\lambda & 10 \\ -1 & -4-\lambda \end{bmatrix} = 0 = 10 + (\lambda-3)(4+\lambda) = \lambda^2 + \lambda - 2$

$$\lambda_1 = -2 \quad \begin{bmatrix} 5 & 10 \\ -1 & -2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

$$\lambda_2 = 1 \quad \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -1 \end{Bmatrix}$$

Modal matrix $M = \begin{bmatrix} 2 & 5 \\ -1 & -1 \end{bmatrix}, \quad M^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -5 \\ 1 & 2 \end{bmatrix}$

$$M^{-1} \frac{d}{dt} \begin{Bmatrix} x \\ y \end{Bmatrix} + M^{-1} \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix} M M^{-1} \begin{Bmatrix} x \\ y \end{Bmatrix} = e^t M^{-1} \begin{Bmatrix} 6 \\ -4 \end{Bmatrix} + M^{-1} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\frac{d}{dt} M^{-1} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} M^{-1} \begin{Bmatrix} x \\ y \end{Bmatrix} = e^t \frac{1}{3} \begin{Bmatrix} 14 \\ -2 \end{Bmatrix} + \frac{1}{3} \begin{Bmatrix} 4 \\ -1 \end{Bmatrix}$$

Let $\begin{Bmatrix} u \\ v \end{Bmatrix} = M^{-1} \begin{Bmatrix} x \\ y \end{Bmatrix}$, then $\frac{du}{dt} - 2u = e^t \frac{14}{3} + \frac{4}{3}, \quad \frac{dv}{dt} + v = -e^t \frac{2}{3} - \frac{1}{3}$

$$u = c_1 e^{2t} - e^t \frac{14}{3} - \frac{2}{3}, \quad v = c_2 e^{-t} - e^t \frac{1}{3} - \frac{1}{3}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = M \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad x = 2u + 5v, \quad y = -u - v$$

State Equation

$$\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{u} \rightarrow e^{-At} \frac{d\bar{x}}{dt} - A e^{-At} \bar{x} = e^{-At} B\bar{u} \rightarrow \frac{d}{dt} (e^{-At} \bar{x}) = e^{-At} B\bar{u}$$

$$\rightarrow \int_0^t \frac{d}{d\tau} (e^{-A\tau} \bar{x}) d\tau = \int_0^t e^{-A\tau} B\bar{u} d\tau \rightarrow \bar{x}(t) = e^{At} [\bar{x}(0) + \int_0^t e^{-A\tau} B\bar{u} d\tau]$$

Another approach

$$\frac{d}{dt} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} -3 & -10 \\ 1 & 4 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + e^t \begin{Bmatrix} 6 \\ -4 \end{Bmatrix} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$M = \begin{bmatrix} 2 & 5 \\ -1 & -1 \end{bmatrix}, \quad M^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -5 \\ 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & -10 \\ 1 & 4 \end{bmatrix} = M \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} M^{-1},$$

$$e^{At} = M \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} M^{-1},$$

$$\int_0^t e^{-A\tau} \left(e^{\tau} \begin{Bmatrix} 6 \\ -4 \end{Bmatrix} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \right) d\tau =$$

$$M \int_0^t \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{2\tau} \end{bmatrix} d\tau M^{-1} \begin{Bmatrix} 6 \\ -4 \end{Bmatrix} + M \int_0^t \begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix} d\tau M^{-1} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= \frac{1}{3} M \begin{bmatrix} 1 - e^{-t} & 0 \\ 0 & 0.5(e^{2t} - 1) \end{bmatrix} \begin{Bmatrix} 14 \\ -2 \end{Bmatrix} + \frac{1}{3} M \begin{bmatrix} 0.5(1 - e^{-2t}) & 0 \\ 0 & e^t - 1 \end{bmatrix} \begin{Bmatrix} 4 \\ -1 \end{Bmatrix}$$

$$= \frac{1}{3} M \begin{bmatrix} 14(1 - e^{-t}) + 2(1 - e^{-2t}) & 0 \\ 0 & -(e^{2t} - 1) - e^t + 1 \end{bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}(t) = M \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \left(M^{-1} \begin{Bmatrix} x \\ y \end{Bmatrix}(0) + \frac{1}{3} \begin{bmatrix} 14(1 - e^{-t}) + 2(1 - e^{-2t}) & 0 \\ 0 & -(e^{2t} - 1) - e^t + 1 \end{bmatrix} \right)$$

Consider $\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{u}$

Taking Laplace transform on both sides yields

$$sL(\bar{x}) - \bar{x}(0) = AL(\bar{x}) + BL(\bar{u}) \rightarrow (sI - A)L(\bar{x}) = \bar{x}(0) + BL(\bar{u}) \rightarrow$$

$$L(\bar{x}) = (sI - A)^{-1} \bar{x}(0) + (sI - A)^{-1} BL(\bar{u}), \quad L^{-1}[(sI - A)^{-1}] = e^{At},$$

$$\rightarrow \bar{x}(t) = e^{At} \bar{x}(0) + \int_0^t e^{A(t-\tau)} B \bar{u}(\tau) d\tau$$

Theorem. If A is a nonsingular matrix, the linear system $A\bar{x} = \bar{b}$ has exactly one solution.

Theorem. A homogeneous system of n linear equations in n unknowns $A\bar{x} = \bar{0}$ has a nontrivial solution if and only if $\det(A) = 0$.

Theorem. If A is a singular matrix of order n such that at least one entry in the k th row of $\det(A)$ has a nonzero cofactor, then a complete solution of the linear system $A\bar{x} = \bar{0}$ is given by the column vector

$$\bar{x} = c[A_{k1}A_{k2} \cdots A_{kn}]^T, \quad c \neq 0$$

Ex. Solve

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 5 \\ 5 & 6 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Ans.

$$A_{11} = \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = -24, \quad A_{12} = -\begin{vmatrix} 3 & 5 \\ 5 & 3 \end{vmatrix} = 16, \quad A_{13} = \begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} = 8$$

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = -24 + 32 - 8 = 0$$

$$x = c[-24 \ 16 \ 8]^T \quad \text{or} \quad x = c[-3 \ 2 \ 1]^T$$

Theorem.

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1 \dots \lambda_n$. Suppose that there exists n linearly independent eigenvectors $V_1 \dots V_n$ associated with these eigenvalues, respectively. Then there exists a matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \equiv D$$

where $P = [V_1 \cdots V_n]$

$$\times \quad A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \text{ has three eigenvalues } \lambda_1 = 1, \lambda_2 = 1, \text{ and } \lambda_3 = -3.$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 & 4 \\ 12 & -11-\lambda & 12 \\ 4 & -4 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 1-\lambda & 12 \\ \lambda-1 & 1-\lambda & 5-\lambda \end{vmatrix}$$

$$= (\lambda-1)^2 \begin{vmatrix} -1 & 0 & 4 \\ 0 & 1 & 12 \\ 1 & 1 & 5-\lambda \end{vmatrix} = (\lambda-1)^2 \begin{vmatrix} -1 & 0 & 4 \\ 0 & 1 & 12 \\ 0 & 1 & 9-\lambda \end{vmatrix} = (\lambda-1)^2(\lambda-3) = 0$$

Matrix

$$\lambda_3 = -3: (A - \lambda_3 I)\bar{x} = \begin{bmatrix} 8 & -4 & 4 \\ 12 & -8 & 12 \\ 4 & -4 & 8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 4 \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = -1, \quad \begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = -3, \quad \begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} = -1,$$

$$\bar{x} = \{-1 -3 -1\}^T \quad \text{or} \quad \bar{x} = \{1 \ 3 \ 1\}^T$$

$$\lambda_1 = \lambda_2 = 1: (A - \lambda_1 I)\bar{x} = \begin{bmatrix} 4 & -4 & 4 \\ 12 & -12 & 12 \\ 4 & -4 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 4 \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Or

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \alpha + \beta \\ \beta \end{Bmatrix}, \quad \text{take} \quad \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \quad \text{L. I.}$$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

§ Orthogonal matrices: $Q \in R^{n \times n}$, $QQ^T = I$

Each column vector of Q is orthonormal, i.e.,

$$q_i^T q_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

§ Unitary matrices: $U \in C^{n \times n}$, $\bar{U}U^T = I$, $\bar{U}$ is the conjugate of U .

Each column vector of U is orthonormal

$$\bar{u}_i^T u_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Consider $\bar{X}' = A\bar{X}$, $A \in R^{n \times n}$.

Let $\bar{X}(t) = e^{\lambda t} \bar{C}$, then $(A - \lambda I)\bar{C}e^{\lambda t} = \bar{0}$ or $(A - \lambda I)\bar{C} = \bar{0}$

Theorem. Let A have eigenvalues $\lambda_1 \dots \lambda_n$. Suppose there are n L.I. eigenvectors

$\vec{E}_1 \dots \vec{E}_n$, respectively, corresponding to these eigenvalues. Then

$\vec{E}_1 e^{\lambda_1 t} \dots \vec{E}_n e^{\lambda_n t}$ are n L.I. solutions. The general solution is expressed as

$$\vec{X}(t) = c_1 \vec{E}_1 e^{\lambda_1 t} + \dots + c_n \vec{E}_n e^{\lambda_n t} \text{ or } \vec{X}(t) = [\Omega(t)]\vec{C}$$

Ex. $\vec{X}' = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \vec{X}$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 6 = \lambda^2 - 7\lambda + 6 = 0$$

$$\lambda_1 = 1: \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 3 \end{Bmatrix}$$

$$\lambda_2 = 6: \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\vec{X}(t) = c_1 \begin{Bmatrix} -2 \\ 3 \end{Bmatrix} e^t + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{6t} \text{ or } \vec{X}(t) = \begin{bmatrix} -2e^t & e^{6t} \\ 3e^t & e^{6t} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

※EX. $\vec{X}' = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix} \vec{X}$

$$\det \begin{bmatrix} 1-\lambda & 3 \\ -3 & 7-\lambda \end{bmatrix} = (1-\lambda)(7-\lambda) + 9 = \lambda^2 - 8\lambda + 16 = 0$$

$$\lambda_1 = 4: \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \vec{X}_1(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{4t}$$

$$\lambda_2 = 4: \text{Try } \vec{X}_2(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} t e^{4t} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} e^{4t}, \text{ then}$$

$$\vec{X}_2'(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{4t} + 4 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} t e^{4t} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} e^{4t} = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix} \left(\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} t e^{4t} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} e^{4t} \right)$$

$$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{4t} + 4 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} e^{4t} = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} e^{4t} \text{ or } \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1-4 & 3 \\ -3 & 7-4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\text{Take } \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \text{ then } \vec{X}_2(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} t e^{4t} + \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{4t}$$

Matrix

$$\begin{aligned}\bar{X}(t) &= c_1 \bar{X}_1(t) + c_2 \bar{X}_2(t) = c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{4t} + c_2 \left(\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} t e^{4t} + \frac{1}{3} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{4t} \right) \\ &= c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{4t} + c_2 \begin{Bmatrix} t+1/3 \\ t+2/3 \end{Bmatrix} e^{4t} = e^{4t} \begin{bmatrix} 1 & t+1/3 \\ 1 & t+2/3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}\end{aligned}$$

Ex. $\begin{Bmatrix} x \\ y \end{Bmatrix}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$ or $y' = y, \quad x' = x + y$

$$y = c_2 e^t, \quad x' = x + c_2 e^t \quad \text{or} \quad x = c_1 e^t + c_2 t e^t,$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}(t) = c_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^t + c_2 \begin{Bmatrix} t \\ 1 \end{Bmatrix} e^t = \begin{bmatrix} e^t & t e^t \\ 0 & e^t \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

Another method

Eigenvalue $\lambda_1 = 1 \quad \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}_1, \quad \begin{Bmatrix} x \\ y \end{Bmatrix}_1(t) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^t,$

Try $\begin{Bmatrix} x \\ y \end{Bmatrix}_2(t) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} e^t,$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_2' = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} (t e^t + e^t) + \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} e^t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} e^t \right)$$

$$\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^t + \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} e^t = \begin{Bmatrix} \alpha + \beta \\ \beta \end{Bmatrix} e^t, \quad \beta = 1, \text{ take } \alpha = 0$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_2(t) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} e^t = \begin{Bmatrix} t e^t \\ e^t \end{Bmatrix}, \quad \begin{Bmatrix} x \\ y \end{Bmatrix}(t) = c_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} e^t + c_2 \begin{Bmatrix} t e^t \\ e^t \end{Bmatrix} = \begin{bmatrix} e^t & t e^t \\ 0 & e^t \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

Ex, $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}, \quad \bar{X}' = A \bar{X}$

A has $\lambda_1 = 1, \quad \bar{E}_1 = \{1 \ 1 \ 0\}^T, \quad \lambda_2 = 1, \quad \bar{E}_2 = \{0 \ 1 \ 1\}^T, \quad \lambda_3 = -3, \quad \bar{E}_3 = \{1 \ 3 \ 1\}^T$

$$\bar{X}(t) = c_1 \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} e^t + c_2 \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} e^t + c_3 \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix} e^{-3t} \quad \text{or} \quad \bar{X}(t) = \begin{bmatrix} e^t & 0 & e^{-3t} \\ e^t & e^t & 3e^{-3t} \\ 0 & e^t & e^{-3t} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

Another method

$$\lambda_1 = 1: \bar{E}_1 = \{1 \ 1 \ 0\}^T, \quad \bar{X}_1(t) = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} e^t, \quad \lambda_3 = -3: \bar{E}_3 = \{1 \ 3 \ 1\}^T, \quad \bar{X}_3(t) = \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix} e^{-3t}$$

$$\lambda_2 = 1: \bar{X}_2(t) = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^t$$

$$\bar{X}_2'(t) = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} e^t + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^t = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \left(\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^t \right)$$

$$\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{bmatrix} 5-1 & -4 & 4 \\ 12 & -11-1 & 12 \\ 4 & -4 & 5-1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{or} \quad \frac{1}{4} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \text{has no answer.}$$

$$\text{So, } \bar{X}_2(t) = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} t e^t + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^t \quad \text{does not exist.}$$

$$\text{EX. } \bar{X}' = A\bar{X}, \quad A = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} -2-\lambda & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 0 & -2-\lambda \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= -(2+\lambda) \begin{vmatrix} 1 & 0 & 1 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = -(2+\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 25 & -7-\lambda & -25 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= -(2+\lambda)[25 - (7+\lambda)(3-\lambda)] = -(2+\lambda)[4 + 4\lambda + \lambda^2] = (\lambda+2)^3$$

Matrix

$$\lambda_1 = -2 : \bar{O} = (A - \lambda_1 I) \bar{X} = \begin{bmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix},$$

$$\bar{X}_1(t) = \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} e^{-2t}, \text{ Try } \bar{X}_2(t) = \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} t e^{-2t} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^{-2t}$$

$$\bar{X}'_2(t) = \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} (t e^{-2t} - 2t e^{-2t}) + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 (-2e^{-2t}) = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} \left(\begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} t e^{-2t} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 e^{-2t} \right)$$

$$\begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} e^{-2t} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 (-2e^{-2t}) = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 e^{-2t}$$

$$\begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2, \quad \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 5 \\ 5 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2, \quad \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 5 \\ 5 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 = \begin{Bmatrix} \alpha \\ 5\alpha - 1 \\ -\alpha \end{Bmatrix}, \text{ Take } \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 4 \\ -1 \end{Bmatrix}, \quad \bar{X}_2(t) = \begin{Bmatrix} 1+t \\ 4+5t \\ -1-t \end{Bmatrix} e^{-2t}$$

$$\bar{X}_3(t) = \frac{1}{2} \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} t^2 e^{-2t} + \begin{Bmatrix} 1 \\ 4 \\ -1 \end{Bmatrix} t e^{-2t} + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 e^{-2t}$$

$$\bar{X}'_3(t) = \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} (t - t^2) e^{-2t} + \begin{Bmatrix} 1 \\ 4 \\ -1 \end{Bmatrix} (1 - 2t) e^{-2t} - 2 \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 e^{-2t}$$

$$= \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} \left(\frac{1}{2} \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} t^2 + \begin{Bmatrix} 1 \\ 4 \\ -1 \end{Bmatrix} t + \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 \right) e^{-2t}$$

$$\begin{Bmatrix} 1 \\ 4 \\ -1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} -1 \\ -1/25 \\ 0 \end{Bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} -1 \\ -1/25 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} -\beta - 1/25 \\ -5\beta - 1 \\ \beta \end{Bmatrix} \quad \text{take } \beta = 1$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_3 = \begin{Bmatrix} -26/25 \\ -6 \\ 1 \end{Bmatrix}, \quad \bar{X}_3(t) = \begin{Bmatrix} t^2/2 + t - 26/25 \\ 5t^2/2 + 4t - 6 \\ -t^2/2 - t + 1 \end{Bmatrix} e^{-2t}$$

$$\bar{X}(t) = c_1 \begin{Bmatrix} t^2/2 + t - 26/25 \\ 5t^2/2 + 4t - 6 \\ -t^2/2 - t + 1 \end{Bmatrix} e^{-2t} + c_2 \begin{Bmatrix} 1+t \\ 4+5t \\ -1-t \end{Bmatrix} e^{-2t} + c_3 \begin{Bmatrix} 1 \\ 5 \\ -1 \end{Bmatrix} e^{-2t}$$

✂ Putzer algorithm for an exponential matrix

Let the eigenvalues of A be $\lambda_1 \bullet \bullet \lambda_n$. Let $f_1(t) \bullet \bullet f_n(t)$ be the unique solutions of the initial value problems $f'_j(t) = \lambda_j f_j(t)$, $f_j(0) = 1$ and

$$f'_j(t) = f_{j-1}(t) + \lambda_j f_j(t), \quad f_j(0) = 0 \quad \text{for } j = 2 \bullet \bullet n,$$

Define n matrices $F_0 \bullet \bullet F_n$ by $F_0 = I_n$ and $F_j = (A - \lambda_1 I_n) \bullet \bullet (A - \lambda_j I_n)$

$$\text{Then } e^{At} = \sum_{j=0}^{n-1} f_{j+1}(t) F_j$$

$$\text{Ex. } A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad f'_1(t) = 1 f_1(t), \quad f_1(t) = e^t, \quad f'_2(t) = e^t + 2 f_2(t), \quad f_2(t) = -e^t + e^{2t}$$

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_1 = A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{2t} - e^t) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$$\text{Ex. } A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad f'_1(t) = 2 f_1(t), \quad f_1(t) = e^{2t}, \quad f'_2(t) = e^{2t} + 2 f_2(t), \quad f_2(t) = t e^{2t}$$

$$F_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_1 = A - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

Matrix

$$e^{At} = e^{2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$ has $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -3$.

$$f'_1 = f_1 \rightarrow f_1 = e^t, \quad f'_2 = f_1 + f_2 \rightarrow f_2 = te^t, \quad f'_3 = f_2 - 3f_3$$

$$f_3 = \frac{1}{4}te^t + \frac{1}{16}(e^{-3t} - e^t)$$

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 4 & -4 & 4 \\ 12 & -12 & 12 \\ 4 & -4 & 4 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -16 & 16 & -16 \\ -48 & 48 & -48 \\ -16 & 16 & -16 \end{bmatrix}$$

$$e^{At} = \sum_{j=0}^2 f_{j+1}(t) F_j$$

Ex. $\bar{X}' = A\bar{X} + \bar{F}(t)$

Let $\bar{X}(t) = e^{At} \bar{U}(t) \rightarrow \bar{X}'(t) = Ae^{At} \bar{U}(t) + e^{At} \bar{U}(t)' = Ae^{At} \bar{U}(t) + \bar{F}(t)$

$$\bar{U}(t)' = e^{-At} \bar{F}(t) \rightarrow \bar{U}(t) = \int e^{-At} \bar{F}(t) dt + \bar{C}, \quad \bar{X}(t) = e^{At} \int e^{-At} \bar{F}(t) dt + \bar{C}e^{At}$$

Ex.

$$\bar{X}' = \begin{bmatrix} 1 & -10 \\ -1 & 4 \end{bmatrix} \bar{X} + \begin{Bmatrix} t \\ 1 \end{Bmatrix}, \quad \bar{X}(0) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad A = \begin{bmatrix} 1 & -10 \\ -1 & 4 \end{bmatrix}$$

$$0 = \begin{vmatrix} 1-\lambda & -10 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1)$$

$$\lambda = -1, \quad \begin{bmatrix} 2 & -10 \\ -1 & 5 \end{bmatrix} \bar{X} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \bar{X} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix}, \quad \bar{X}(t) = \begin{Bmatrix} 5e^{-t} \\ e^{-t} \end{Bmatrix}$$

$$\lambda = 6, \quad \begin{bmatrix} -5 & -10 \\ -1 & -2 \end{bmatrix} \bar{X} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \bar{X} = \begin{Bmatrix} -2 \\ 1 \end{Bmatrix}, \quad \bar{X} = \begin{Bmatrix} -2e^{6t} \\ e^{6t} \end{Bmatrix}, \quad e^{At} = \begin{bmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{bmatrix},$$

$$e^{-At} = \frac{1}{7} \begin{bmatrix} e^t & 2e^t \\ -e^{-6t} & 5e^{-6t} \end{bmatrix}, \quad e^{-At} \begin{Bmatrix} t \\ 1 \end{Bmatrix} = \frac{1}{7} \begin{bmatrix} e^t & 2e^t \\ -e^{-6t} & 5e^{-6t} \end{bmatrix} \begin{Bmatrix} t \\ 1 \end{Bmatrix} = \frac{1}{7} \begin{Bmatrix} (2+t)e^t \\ (5-t)e^{-6t} \end{Bmatrix}$$

Matrix

$$\int e^{-At} \begin{Bmatrix} t \\ 1 \end{Bmatrix} dt = \frac{1}{7} \begin{Bmatrix} (1+t)e^t \\ (t-29/6)e^{-6t} / 6 \end{Bmatrix},$$

$$e^{At} \int e^{-At} \begin{Bmatrix} t \\ 1 \end{Bmatrix} dt = \frac{1}{7} \begin{bmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{bmatrix} \begin{Bmatrix} (1+t)e^t \\ (t-29/6)e^{-6t} / 6 \end{Bmatrix} = \frac{1}{18} \begin{Bmatrix} 84t+119 \\ 21t+21 \end{Bmatrix}$$

$$\bar{X}(t) = \frac{1}{18} \begin{Bmatrix} 84t+119 \\ 21t+21 \end{Bmatrix} + \frac{1}{7} \begin{bmatrix} e^t & 2e^t \\ -e^{-6t} & 5e^{-6t} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\bar{X}(0) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{1}{18} \begin{Bmatrix} 119 \\ 21 \end{Bmatrix} + \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}, \quad \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = -\frac{1}{18} \begin{Bmatrix} 499 \\ 104 \end{Bmatrix}$$

$$\bar{X}(t) = \frac{1}{18} \begin{Bmatrix} 84t+119 \\ 21t+21 \end{Bmatrix} - \frac{1}{126} \begin{Bmatrix} 707e^t \\ 21e^{-6t} \end{Bmatrix}$$

Laplace transform $L(\bar{X}') = L[A\bar{X} + \bar{F}(t)] \rightarrow (sI - A)L(\bar{X}) = L(\bar{F}) + \bar{X}(0)$

$$L(\bar{X}) = (sI - A)^{-1} L(\bar{F}) + (sI - A)^{-1} \bar{X}(0) \rightarrow \bar{X}(t) = \int_0^t e^{A(t-\tau)} \bar{F}(\tau) d\tau + e^{At} \bar{X}(0)$$

$$\int_0^t e^{A(t-\tau)} \begin{Bmatrix} \tau \\ 1 \end{Bmatrix} d\tau = \int_0^t \begin{bmatrix} 5e^{-(t-\tau)} & -2e^{6(t-\tau)} \\ e^{-(t-\tau)} & e^{6(t-\tau)} \end{bmatrix} \begin{Bmatrix} \tau \\ 1 \end{Bmatrix} d\tau$$